SUBFAIR CASINO FUNCTIONS ARE SUPERADDITIVE

BY

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ABSTRACT

It is shown that all subfair casino functions are superadditive on the unit interval.

As the main step in showing that bold play in a primitive casino is optimal, and, consequently, that the utility U of the bold strategies in a primitive casino is a casino function, it was shown in [1, Chap. 6] that for all f and g in the closed unit interval, U satisfies:

$$U(f+g) \geqq U(f) + U(g) \quad \text{if } f+g \leqq 1, \tag{1}$$

and

$$U(f+g-1) \geqq U(f) + U(g) - 1 \quad \text{if } f+g \geqq 1. \tag{2}$$

It turns out to be very simple to prove

THEOREM 1. (**) *Every subfair casino function U satisfies* (1) *and* (2).

Proof.* We first show that U satisfies (1). As was observed in [1, Chap. 4, Secs. 2 and 3],

$$U(fg) \geqq U(f)U(g), \tag{3}$$

and

$$U(f+g(1-f)) \geqq U(f) + U(g)(1-U(f)), \tag{4}$$

for all f and g in the unit interval.

The only discontinuous subfair casino function is 0 for $0 \leqq f < 1$ and is 1 at $f=1$, as was shown in [1, Chap. 4, Sec. 3]. Since this function plainly satisfies (1) and (2), only continuous U need to be considered. Let

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$$S(f,g) = U(f) + U(g) - U(f+g) \tag{5}$$

on the triangle where $f + g \leqq 1$.

If S were ever positive, there is a least f such that S attains its maximum for that f and some g. That f cannot be 0, nor can a corresponding g be 1. Apply S to (f', g') where $f' = fg$ and $g' = f + g - fg$.

$$\begin{aligned} S(f',g') &= U(f') + U(g') - U(f'+g') \\ &= U(fg) + U(f+g-fg) - U(f+g) \\ &\geqq U(f)U(g) + U(f) + U(g) - U(f)U(g) - U(f+g) \qquad (6) \\ &= U(f) + U(g) - U(f+g) \\ &= S(f,g), \end{aligned}$$

where the inequality is an application of (3) and (4). But (6) is a contradiction, since $f' < f$. So (1) holds.

A very similar proof would show that (2) holds, but it seems more interesting to demonstrate (2) by means of a digression that brings out the intimate relation of (2) to (1).

To each function V of two real variables associate its dual V^*, thus.

$$V^*(f,g) = 1 - V(1-f, 1-g). \tag{7}$$

Let $\mathscr{V}$ be the set of all V such that: (i) $V(f,g)$ is increasing in f and in g; and (ii) for every subfair casino function U,

$$U(V(f,g)) \geqq V(U(f), U(g)) \tag{8}$$

whenever f, g and $V(f,g)$ are in the unit interval.

LEMMA 1. *$V \in \mathscr{V}$ if and only if $V^* \in \mathscr{V}$.*

Proof of Lemma. 1: Introduce U^* as in [1, Chap. 4, Sec. 6], namely, $U^*(z) = 1 - U^{-1}(1-z)$ for $0 \leqq z \leqq 1$. Let f, g, x and y be related thus. $U(f) = 1 - x$, $U(g) = 1 - y$, or, equivalently, $U^*(x) = 1 - f$, $U^*(y) = 1 - g$. Let $V \in \mathscr{V}$ and suppose first that $V(f,g)$ and $V^*(x\ y)$ are in the unit interval. Then notice that (8) holds if and only if

$$U^*(V^*(x,y)) \geqq V^*(U^*(x), U^*(y)), \tag{9}$$

even if V is not monotone. Therefore, it may be supposed that x, y, and $V^*(x,y)$ are in the unit interval, but $V(f,g)$ is not. Verify that because $V^*(x,y)$ is in the unit interval, so is $V(U(f), U(\text{g}))$. So by monotoneity of V, $V(f,g)$ cannot be less than 0, and therefore must exceed 1. So $V^*(U^*(x), U^*(y)) = 1 - V(f,g) < 0 \leqq U^*(V^*(x,y))$. This completes the proof of Lemma 1.

Since (1) has been shown to hold for all subfair casino functions, the function

$V(f, g) = f + g$ is in $\mathscr{V}$. In view of Lemma 1, so is the function $f + g - 1$, that is, (2) also holds for all subfair casino functions U. The proof of Theorem 1 is now complete.

As is easily seen, the proofs of Theorem 1 and Lemma 1 apply not only to casino functions U but to all bounded solutions U to (3) and (4) omitting those U for which $U(f) \equiv 1$ for $0 < f < 1$.

Inequalities (3) and (4) for casino functions U have intuitive interpretations that make them apriori plausible and, therefore, natural to conjecture. It would be nice to find such interpretations for (1) and (2).

References

1. L. E. Dubins, and L. J. Savage, *How To Gamble If You Must*. McGraw-Hill, New York. 1965.

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